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# On the Positive Almost Periodic Solutions of a Class of Lotka–Volterra Type Systems with Delays

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In this paper we study the existence of positive almost periodic solutions for a class of almost periodic Lotka–Volterra type systems with delays. Applying Schauder's fixed point theorem we obtain a general criterion of the existence of positive almost periodic solutions. This criterion can be used not only in the case of finite delay but also in the case of infinite delay. © 2000 Academic Press

*Key Words:* Lotka–Volterra type system; delay; almost periodic solution; Schauder's fixed point theorem; global asymptotical stability.

## 1. INTRODUCTION

Let $R = (-\infty, \infty)$ and let $R^n$ be $n$-dimensional Euclidean space. Let $C^n[-\tau, 0]$ denote the Banach space of continuous bounded functions $\phi$: $[-\tau, 0] \to R^n$ with the supremum norm $\|\phi\| = \max\{|\phi(s)| : s \in [-\tau, 0]\}$, where $\phi = (\phi_1, \phi_2, \dots, \phi_n)$, $|\phi(s)| = \sum_{i=1}^{n} |\phi_i(s)|$, and $\tau$ is a nonnegative constant or $\tau = +\infty$. Let $F(t, \phi)$: $R \times C^n[-\tau, 0] \to R^n$ be a continuous real functional, $F(t, \phi) = (f_1(t, \phi), f_2(t, \phi), \dots, f_n(t, \phi))$. We assume that (a) $F(t, \phi)$ is uniformly almost periodic with respect to $t \in R$ for $\phi \in C^n[-\tau, 0]$; i.e., for any given constant $\varepsilon > 0$ and bounded closed set $S$ in $C^n[-\tau, 0]$ there is a constant $l = l(\varepsilon, S) > 0$ such that $[t, t + l] \cap T(F, \varepsilon, S) \neq \varnothing$ for any $t \in R$, where $T(F, \varepsilon, S) = \{u \in R : |F(t + u, \phi) - F(t, \phi)| < \varepsilon$ for all $(t, \phi) \in R \times S\}$; (b) $F(t, \phi)$ is locally Lipschitz with respect to $\phi \in C^n[-\tau, 0]$; i.e., for any constant $M > 0$ there is constant $L = L(M) > 0$ such that $|F(t, \phi_1) - F(t, \phi_2)| \leq L\|\phi_1 - \phi_2\|$ for all $t \in R$, $\phi_1, \phi_2 \in C^n[-\tau, 0]$, and $\|\phi_1\| \leq M$, $\|\phi_2| \leq M$; (c) $F(t, 0) \equiv 0$ for all $t \in R$.

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We consider the almost periodic differential equation with delay,

$$\frac{dx_i(t)}{dt} = x_i(t)\big(a_i(t) - b_i(t)x_i(t) - f_i(t, x_t)\big), \qquad i = 1, 2, \dots, n, \quad (1)$$

where $t \in R$, $x(t) = (x_1(t), x_2(t), \dots, x_n(t)) \in R^n$, $x_t = (x_{1t}, x_{2t}, \dots, x_{nt}) \in C^n[-\tau, 0]$, and $x_{it}(s) = x_i(t+s)$ $(i = 1, 2, \dots, n)$ for all $s \in [-\tau, 0]$; $a_i(t)$ and $b_i(t)$ $(i = 1, 2, \dots, n)$ are continuous almost periodic functions with respect to $t \in R$ and $b_i(t) \geq 0$ for all $t \in R$.

In the theory of mathematical biology Eq. (1) is generally called a Lotka–Volterra type system with delay. It is very important in the models of multi-species population dynamics. The assumption of almost periodicity of system (1) is a way of incorporating the time dependent variability of the environment, especially when the various components of the environment are periodic with not necessarily commensurate periods. Mathematically, system (1) will denote a generalization of an autonomous and a periodic system. For some special cases of system (1) the existence and globally asymptotic stability of positive almost periodic solutions have been studied in many works, for example, [2, 7, 8]. Here, a solution of system (1) is said to be positive almost periodic if its all components are almost periodic functions which are bounded above and below by positive constants on $R$. In this paper our purpose is to study the existence of positive almost periodic solutions directly for the general cases of system (1) by means of Schauder's fixed point theorem [4]. We shall establish a general criterion of the existence of positive almost periodic solutions. We shall see that the conditions required in this criterion are quite weak. It can be applicable not only in the case of finite delay, but also in the case of infinite delay for system (1). In many special cases, this criterion can be easily checked and reduced to some well known results.

## 2. ALMOST PERIODIC LODISTIC EQUATION

For any almost periodic function $f(t)$: $R \to R^m$, where $R^m$ is $m$-dimensional Euclidean space, we denote the hull of $f(t)$ by $H(f(t))$, the module of $f(t)$ by $\operatorname{mod}(f(t))$, and the mean of $f(t)$ by $m(f(t))$. The explanations in detail about these notations can be found in [5, 6].

Consider the almost periodic logistic equation

$$\frac{du}{dt} = u(\alpha(t) - \beta(t)u), \quad (2)$$

where $\alpha(t)$ and $\beta(t)$ are continuous almost periodic functions defined on $R$, $\beta(t) \geq 0$ for all $t \in R$.

We say that a solution $u(t)$ of Eq. (2) is positive almost periodic if $u(t)$ is an almost periodic function defined on $R$ and $0 < \inf\{u(t) : t \in R\} \le \sup\{u(t) : t \in R\} < \infty$. First of all, we have the following result.

LEMMA 1. *Assume* $m(\alpha(t)) > 0$ *and* $m(\beta(t)) > 0$; *then Eq.* (2) *has a unique positive almost periodic solution* $u^*(t)$ *and* $\operatorname{mod}(u^*(t)) \subset \operatorname{mod}(\alpha(t), \beta(t))$.

*Proof.* For any $(\alpha^*(t), \beta^*(t)) \in H(\alpha(t), \beta(t))$ we have the following hull equation of Eq. (2):

$$\frac{dv}{dt} = v(\alpha^*(t) - \beta^*(t)v). \tag{3}$$

By $m(\alpha(t)) > 0$ and $m(\beta(t)) > 0$ we obtain that there are positive constants $\delta$ and $\omega$ such that

$$\int_t^{t+\omega} \alpha^*(s)\,ds > \delta, \qquad \int_t^{t+\omega} \beta^*(s)\,ds > \delta \qquad \text{for all } t \in R.$$

Hence, there are positive constants $k_1$, $k_2$, $\varepsilon_0$ and $k_1 > k_2$ such that

$$\begin{aligned} &\int_t^{t+\omega} (\alpha^*(s) - \beta^*(s)k_1)\,ds < -\varepsilon_0, \\ &\int_t^{t+\omega} (\alpha^*(s) - \beta^*(s)k_2)\,ds > \varepsilon_0, \end{aligned} \tag{4}$$

for all $t \in R$. Let $a_i = \sup\{|\alpha(t)| + \beta(t)k_i : t \in R\}$ $(i = 1, 2)$; then $a_i > 0$. For any integer $m > 0$, consider the solution $v_m(t)$ of Eq. (3) with initial value $v_m(-m) \in (k_2, k_1)$. We first prove that

$$v_m(t) \le k_1 \exp(a_1\omega), \qquad \text{for all } t \ge -m. \tag{5}$$

In fact, if the inequality (5) does not hold, then from $v_m(-m) < k_1 < k_1 \exp(a_1\omega)$ there are $t_2 > t_1 > -m$ such that $v_m(t_2) > k_1 \exp(a_1\omega)$, $v_m(t_1) = k_1$, and $v_m(t) > k_1$ for all $t \in (t_1, t_2]$. Choose the integer $p \ge 0$ such that $t_2 \in (t_1 + p\omega, t_1 + (p+1)\omega]$; then in view of (4) we have

$$\begin{aligned} k_1 \exp(a_1\omega) &< v_m(t_2) \\ &= v_m(t_1)\exp\int_{t_1}^{t_2} (\alpha^*(t) - \beta^*(t)v_m(t))\,dt \\ &\le k_1 \exp\left[\int_{t_1}^{t_1+p\omega} + \int_{t_1+p\omega}^{t_2} (\alpha^*(t) - \beta^*(t)k_1)\,dt\right] \\ &\le k_1 \exp\int_{t_1+p\omega}^{t_2} (\alpha^*(t) - \beta^*(t)k_1)\,dt \\ &\le k_1 \exp(a_1\omega), \end{aligned}$$

which is a contradiction. Similarly, we can prove that

$$v_m(t) \geq k_2 \exp(-a_2 \omega), \qquad \text{for all } t \geq -m. \tag{6}$$

Consider the sequence $\{v_m(t)\}$ of solutions of Eq. (3). The inequality (5) implies that there is constant $M > 0$ such that

$$\left| \frac{dv_m(t)}{dt} \right| \leq M, \qquad \text{for all } t \geq -m, \quad m = 1, 2, \dots .$$

Hence, for any integer $p > 0$ the sequence $\{v_m(t) : m \geq p\}$ is uniformly bounded and equicontinuous on interval $[-p, p]$. Applying the Ascolia–Arzela theorem to sequence $\{v_m(t)\}$ we have that $\{v_m(t)\}$ has a subsequence $\{v_{m1}(t)\}$ that uniformly converges on $[-1, 1]$, $\{v_{m1}(t)\}$ has a subsequence $\{v_{m2}(t)\}$ that uniformly converges on $[-2, 2]$, $\{v_{m2}(t)\}$ has a subsequence $\{v_{m3}(t)\}$ that uniformly converges on $[-3, 3]$, and so on. Choose the diagonal sequence $\{v_{mm}(t)\}$; then we obtain that $\{v_{mm}(t)\}$ uniformly converges on $[-p, p]$ for any integer $p > 0$. Let $v^*(t)$ be the limit function of $\{v_{mm}(t)\}$; obviously, $v^*(t)$ is defined on $R$. In view of (5) and (6) we have

$$k_2 \exp(-a_2 \omega) \leq v^*(t) \leq k_1 \exp(a_1 \omega), \qquad \text{for all } t \in R.$$

It is not difficult to check that $v^*(t)$ satisfies Eq. (3) on $R$. Hence, $v^*(t)$ is a strictly positive solution of Eq. (3). Here a solution of Eq. (3) is said to be strictly positive, if it is bounded above and below by positive constants on $R$.

Now, we prove the uniqueness of a strictly positive solution of Eq. (3). Suppose that $w^*(t)$ is also a strictly positive solution of Eq. (3). Consider the function $V(t) = |\ln v^*(t) - \ln w^*(t)|$. By calculating the Dini upper right derivative of $V(t)$ we have

$$\begin{aligned} D^+ V(t) &\leq -\beta^*(t) |v^*(t) - w^*(t)| \\ &\leq -\beta^*(t) \eta V(t), \qquad \text{for all } t \in R, \end{aligned}$$

where $\eta = \inf\{v^*(t), w^*(t) : t \in R\} > 0$. We have

$$V(t) \geq V(0) \exp \int_t^0 \eta \beta^*(s)\, ds$$

for any $t \leq 0$. Since $\int_{-\infty}^0 \beta^*(t)\, dt = \infty$ and $V(t)$ is bounded on $R$, then $V(0) = 0$; i.e., $v^*(0) = w^*(0)$. By the uniqueness of solutions of the initial value problem for Eq. (3) we obtain that $v^*(t) \equiv w^*(t)$ for all $t \in R$.

Finally, since for any $(\alpha^*(t), \beta^*(t)) \in H(\alpha(t), \beta(t))$ Eq. (3) has a unique strictly positive solution $v^*(t)$, by [6, Theorem 3.2] (of the existence of almost periodic solutions) we obtain that Eq. (2) has a unique positive almost periodic solution $u^*(t)$ and $\operatorname{mod}(u^*(t)) \subset \operatorname{mod}(\alpha(t), \beta(t))$. The proof is complete.

Consider the almost periodic logistic equations

$$\frac{du}{dt} = u(\alpha_1(t) - \beta_1(t)u), \tag{7}$$

$$\frac{dv}{dt} = v(\alpha_2(t) - \beta_2(t)v), \tag{8}$$

where $\alpha_i(t)$ and $\beta_i(t)$ $(i = 1, 2)$ are continuous almost periodic functions defined on $R$. We assume that $\alpha_1(t) \le \alpha_2(t)$ and $\beta_1(t) \ge \beta_2(t) \ge 0$ for all $t \in R$. We have the following result.

LEMMA 2. *Assume* $m(\alpha_1(t)) > 0$ *and* $m(\beta_2(t)) > 0$; *then Eqs.* (7) *and* (8) *have unique positive almost periodic solutions* $u^*(t)$ *and* $v^*(t)$, *respectively, such that* $u^*(t) \le v^*(t)$ *for all* $t \in R$.

*Proof.* The existence and uniqueness of the positive almost periodic solutions $u^*(t)$ and $v^*(t)$ of Eqs. (7) and (8) is assured by Lemma 1. Choose positive constants $\omega$, $k_1$, $k_2$, and $\varepsilon_0$ such that

$$\begin{aligned} &\int_t^{t+\omega} (\alpha_1(s) - \beta_1(s)k_1)\, ds > \varepsilon_0 \\ &\int_t^{t+\omega} (\alpha_2(s) - \beta_2(s)k_2)\, ds < -\varepsilon_0 \end{aligned} \tag{9}$$

for all $t \in R$. For any integer $m > 0$ we consider the solutions $u_m(t)$ and $v_m(t)$ of Eqs. (7) and (8) with initial value $u_m(-m) = v_m(-m) \in (k_1, k_2)$. Then on the basis of (9) and the differential inequality theorem, using the method with which we prove (5) and (6) we can obtain

$$k_1 \exp(-b_1\omega) \le u_m(t) \le v_m(t) \le k_2 \exp(b_2\omega) \tag{10}$$

for all $t \ge -m$, where $b_i = \sup\{|\alpha_i(t)| + \beta_i(t)k_i : t \in R\}$ $(i = 1, 2)$. Applying the same method given in the proof of Lemma 1 it follows that the sequence $\{u_m(t)\}$ has a subsequence $\{u_{m1}(t)\}$ that uniformly converges to the positive almost periodic solution $u^*(t)$ of Eq. (7) on any finite interval in $R$. Similarly, the sequence $\{v_{m1}(t)\}$ has a subsequence $\{v_{m2}(t)\}$ that uniformly converges to the positive almost periodic solution $v^*(t)$ of Eq. (8) on any finite interval in $R$. Finally, in view of (10) we can obtain that $u^*(t) \le v^*(t)$ for all $t \in R$. The proof is complete.

## 3. MAIN RESULTS

We say that a solution $x(t) = (x_1(t), x_2(t), \dots, x_n(t))$ of system (1) is positive almost periodic, if $x(t)$ is an almost periodic function defined on $R$ and $0 < \inf\{x_i(t) : t \in R\} \le \sup\{x_i(t) : t \in R\} < \infty$ for all $i = 1, 2, \dots, n$.

In system (1), we assume that $m(a_i(t)) > 0$ and $m(b_i(t)) > 0$ for $i = 1, 2, \dots, n$. Consider the almost periodic logistic equations

$$\frac{dx_i(t)}{dt} = x_i(t)\big(a_i(t) - b_i(t)x_i(t)\big), \qquad i = 1, 2, \dots, n. \tag{11}$$

By Lemma 1, for each $i = 1, 2, \dots, n$ Eq. (11) has a unique positive almost periodic solution $x_{i0}(t)$ such that $\mathrm{mod}(x_{i0}(t)) \subset \mathrm{mod}(a_i(t), b_i(t))$, respectively.

Let $x_0(t) = (x_{10}(t), x_{20}(t), \dots, x_{n0}(t))$; further we assume that $m(a_i(t) - f_i(t, x_{0t})) > 0$ for $i = 1, 2, \dots, n$. Consider the following almost periodic logistic equations:

$$\frac{dx_i(t)}{dt} = x_i(t)\big(a_i(t) - f_i(t, x_{0t}) - b_i(t)x_i(t)\big), \qquad i = 1, 2, \dots, n. \tag{12}$$

By Lemma 1, for each $i = 1, 2, \dots, n$, Eq. (12) has a unique positive almost periodic solution $x_{ix_0}(t)$ such that $\mathrm{mod}(x_{ix_0}(t)) \subset \mathrm{mod}(a_i(t) - f_i(t, x_{0t}), b_i(t))$, respectively.

By $AP(R, R^n)$ we denote the Banach space of continuous almost periodic functions $\psi$: $R \to R^n$ under the supremum norm $\|\psi\| = \sup\{|\psi(s)| : s \in R\}$. Let $Q(t, \phi) = (q_1(t, \phi), q_2(t, \phi), \dots, q_n(t, \phi))$, where $q_i(t, \phi) = a_i(t) - b_i(t)\phi_i(0) - f_i(t, \phi)$ $(i = 1, 2, \dots, n)$ and $\phi = (\phi_1, \phi_2, \dots, \phi_n) \in C^n[-\tau, 0]$. It is obvious that $Q(t, \phi)$ is uniformly almost periodic with respect to $t \in R$ for $\phi \in C^n[-\tau, 0]$. Let $D = \{\psi : \psi \in AP(R, R^n), \mathrm{mod}(\psi(t)) \subset \mathrm{mod}(Q(t, \phi))\}$; then we can prove that $D$ is also a Banach space with the supremum norm $\|\psi\|$. Obviously, by $\mathrm{mod}(a_i(t), b_i(t)) \subset \mathrm{mod}(Q(t, \phi))$ $(i = 1, 2, \dots, n)$ we obtain $x_0 \in D$. We construct the bounded, closed, and convex set $G$ in $D$ as follows:

$$G = \{\psi = (\psi_1, \psi_2, \dots, \psi_n) \in D : 0 \le \psi_i(s) \le x_{i0}(s),$$
$$s \in R, i = 1, 2, \dots, n\}.$$

Further, we assume that for any $\psi \in G$,

$$0 \le f_i(t, \psi_t) \le f_i(t, x_{0t}), \qquad \text{for all } t \in R, \quad i = 1, 2, \dots, n, \tag{13}$$

where $\psi_t(s) = \psi(t + s)$ and $x_{0t}(s) = x_0(t + s)$ for all $s \in [-\tau, 0]$. For any $\psi \in G$, we consider the following almost periodic logistic equations:

$$\frac{dx_i(t)}{dt} = x_i(t)\big(a_i(t) - f_i(t, \psi_t) - b_i(t)x_i(t)\big), \qquad i = 1, 2, \dots, n. \tag{14}$$

From (13) and Lemma 1 it follows that for each $i = 1, 2, \ldots, n$ Eq. (14) has a unique positive almost periodic solution $x_{i\psi}(t)$ such that $\operatorname{mod}(x_{i\psi}(t)) \subset \operatorname{mod}(a_i(t) - f_i(t, \psi_t), b_i(t))$, respectively. Let $x_\psi(t) = (x_{1\psi}(t), x_{2\psi}(t), \ldots, x_{n\psi}(t))$; obviously we have $\operatorname{mod}(x_\psi(t)) \subset \operatorname{mod}(Q(t, \phi))$. By Lemma 2 we see that $x_{ix_0}(t) \le x_{i\psi}(t) \le x_{i0}(t)$ for all $t \in R$ and $i = 1, 2, \ldots, n$. Hence, we have $x_\psi \in G$.

Now, let us define the map $P$: $G \to G$ as follows:

$$P(\psi) = x_\psi, \qquad \text{for all } \psi \in G.$$

To apply Schauder's fixed point theorem [4] to the map $P$ we need to prove that $P$ is continuous on $G$ and the set $P(G) = \{x_\psi : \psi \in G\}$ is relatively compact in $D$. In fact, for any sequence $\{\psi_k\} \subset G$ such that $\psi_k \to \psi \in G$ in $D$ as $k \to \infty$. By the local Lipschitz property of $F(t, \phi)$ with respect to $\phi \in C^n[-\tau, 0]$ there is the constant $L > 0$ such that

$$\left| F(t, \psi_{kt}) - F(t, \psi_t) \right| \le L \|\psi_{kt} - \psi_t\| \le L \|\psi_k - \psi\| \tag{15}$$

for all $t \in R$ and $k = 1, 2, \ldots$ . From Eq. (14) we directly obtain that the almost periodic solutions $x_{i\psi_k}(t)$ and $x_{i\psi}(t)$ are given by

$$x_{i\psi_k}(t) = \left[ \int_{-\infty}^{t} b_i(s) \exp \int_s^t (f_i(u, \psi_{ku}) - a_i(u))\, du\, ds \right]^{-1}$$

and

$$x_{i\psi}(t) = \left[ \int_{-\infty}^{t} b_i(s) \exp \int_s^t (f_i(u, \psi_u) - a_i(u))\, du\, ds \right]^{-1}$$

for all $t \in R$, $i = 1, 2, \ldots, n$, and $k = 1, 2, \ldots$ . Estimating $|x_{i\psi_k}(t) - x_{i\psi}(t)|$ for each $i = 1, 2, \ldots, n$, then by the mean value theorem and (15) we obtain

$$\begin{aligned}
&\left| x_{i\psi_k}(t) - x_{i\psi}(t) \right| \\
&\quad \le M_i(k) \left| \int_{-\infty}^{t} \left[ \exp \int_s^t (f_i(u, \psi_{ku}) - a_i(u))\, du \right.\right. \\
&\qquad\qquad \left.\left. - \exp \int_s^t (f_i(u, \psi_u) - a_i(u))\, du \right] ds \right| \\
&\quad \le M_i(k) \left| \int_{-\infty}^{t} \int_s^t (f_i(u, \psi_{ku}) - f_i(u, \psi_u))\, du \exp \sigma_i(t, s)\, ds \right| \\
&\quad \le M_i(k) L \|\psi_k - \psi\| \int_{-\infty}^{t} (t - s) \exp \sigma_i(t, s)\, ds
\end{aligned}$$

for all $t \in R$ and $k = 1, 2, \ldots$, where

$$M_i(k) = \sup\left\{ |b_i(t)| x_{i\psi_k}^{-1}(t) x_{i\psi}^{-1}(t) : t \in R \right\}$$
$$\leq \sup\left\{ |b_i(t)| x_{ix_0}^{-2}(t) : t \in R \right\}$$

and $\sigma_i(t, s)$ is situated between $\int_s^t (f_i(u, \psi_{ku}) - a_i(u))\, du$ and $\int_s^t (f_i(u, \psi_u) - a_i(u))\, du$. Since $\sigma_i(t, s) \leq \int_s^t (f_i(u, x_{0u}) - a_i(u))\, du$ for all $t \geq s$ and $m(a_i(t) - f_i(t, x_{0t})) > 0$, we can obtain that there are constants $\lambda_i > 0$ and $H_i > 0$ such that $\exp \sigma_i(t, s) \leq H_i \exp \lambda_i(s - t)$ for all $t \geq s$. Hence, we have

$$\left| x_{i\psi_k}(t) - x_{i\psi}(t) \right| \leq M_i(k) H_i L \|\psi_k - \psi\| \int_{-\infty}^{t} (t - s) \exp \lambda_i(s - t)\, ds$$
$$\leq M_i(k) H_i L \lambda_i^{-2} \|\psi_k - \psi\|$$

for all $t \in R$ and $k = 1, 2, \ldots$ . Therefore, we have that $x_{\psi k} \to x_\psi$ in $D$ as $k \to \infty$. It shows that the map $P$ is continuous on $G$.

It is obvious that $P(G)$ is bounded in $D$. For any $x_\psi = (x_{1\psi}, x_{2\psi}, \ldots, x_{n\psi}) \in P(G)$, since $x_{i\psi}(t)$ is the solution of Eq. (14), $0 < x_{i\psi}(t) \leq x_{i0}(t) \leq \sup\{x_{i0}(t) : t \in R\} < \infty$, and $q_i(t, \phi)$ is uniformly almost periodic with respect to $t \in R$ for $\phi \in C^n[-\tau, 0]$, then from (14) we obtain that there is constant $M > 0$, and $M$ is independent on $x_\psi$, such that

$$\left| \frac{dx_{i\psi}(t)}{dt} \right| \leq M, \qquad \text{for all } t \in R, \quad i = 1, 2, \ldots, n.$$

Hence, $P(G)$ is equicontinuous on $R$. For any sequence $\{\psi_k\}$ in $P(G)$, by the Ascoli–Arzela theorem and in accordance with the same method given in the proof of Lemma 1, we obtain that there is a subsequence of $\{\psi_k\}$, say $\{\psi_k\}$ again, such that it is uniformly convergent on any finite interval in $R$. By the uniformly almost periodicity of $Q(t, \phi)$, we obtain that for any given constant $\varepsilon > 0$ there is a constant $l = l(\varepsilon, G) > 0$ such that $[t, t + l] \cap T(Q, \varepsilon, G) \neq \phi$ for any $t \in R$, where $T(Q, \varepsilon, G) = \{u \in R : |Q(t + u, \phi) - Q(t, \phi)| < \varepsilon$ for all $(t, \phi) \in R \times G\}$. From $\operatorname{mod}(\psi_k(t)) \subset \operatorname{mod}(Q(t, \phi))$, there is a constant $\delta > 0$ and $\delta < \varepsilon$ such that $T(Q, \delta, G) \subset T(\psi_k, \varepsilon)$ for all $k > 0$, where $T(Q, \delta, G)$ is defined to be similar to $T(Q, \varepsilon, G)$ and $T(\psi_k, \varepsilon) = \{u \in R : |\psi_k(t + u) - \psi_k(t)| < \varepsilon$ for all $t \in R\}$. Hence, for any $t \in R$ there is $u \in T(Q, \delta, G)$ such that $t + u \in [0, l]$. Since

$$\left| \psi_k(t) - \psi_{k+p}(t) \right| \leq \left| \psi_k(t) - \psi_k(t + u) \right| + \left| \psi_{k+p}(t) - \psi_{k+p}(t + u) \right|$$
$$+ \left| \psi_{k+p}(t + u) - \psi_k(t + u) \right|,$$

and there is a constant $K(\varepsilon, G) > 0$ such that for any $k > K(\varepsilon, G)$

$$\left|\psi_k(t) - \psi_{k+p}(t)\right| < \varepsilon, \qquad \text{for all } t \in [0, l], \quad p = 1, 2, \dots,$$

then we obtain that $|\psi_k(t) - \psi_{k+p}(t)| < 3\varepsilon$ for all $t \in R$ and $p = 1, 2, \dots$ when $k > K(\varepsilon, G)$. Hence, $\{\psi_k\}$ is uniformly convergent on $R$. It shows that $P(G)$ is relatively compact in $D$.

Applying Schauder's fixed-point theorem we obtain that the map $P$ has a fixed point $\psi_0 \in G$. Let $\psi_0(t) = (\psi_{10}(t), \psi_{20}(t), \dots, \psi_{n0}(t))$; then we have

$$\frac{d\psi_{i0}(t)}{dt} = \psi_{i0}(t)\big(a_i(t) - b_i(t)\psi_{i0}(t) - f_i(t, \psi_{0t})\big), \qquad i = 1, 2, \dots, n.$$

It shows that $\psi_0(t)$ is a positive almost periodic solution of system (1). Thus we have proved the following result.

THEOREM 1. *If* $0 \le f_i(t, \psi_t) \le f_i(t, x_{0t})$ *for any* $\psi \in G$, $t \in R$, *and* $i = 1, 2, \dots, n$, *and*

$$m\big(b_i(t)\big) > 0, \qquad m\big(a_i(t) - f_i(t, x_{0t})\big) > 0, \qquad i = 1, 2, \dots, n,$$

*then system* (1) *has at least a positive almost periodic solution.*

*Remark* 1. The delay $\tau$ can be equal to $\infty$ in the proof of Theorem 1. Therefore, Theorem 1 contains not only the case of finite delay of system (1), but also the case of infinite delay.

## 4. APPLICATION

Let us consider the nonautonomous $N$-spaces Lotka–Volterra type competitive system with distributed delay and discrete delay,

$$\frac{dx_i(t)}{dt} = x_i(t)\Bigg(a_i(t) - b_i(t)x_i(t) - \sum_{j=1}^{n}\sum_{l=1}^{l_{ij}} a_{ijl}(t)x_j\big(t - \tau_{ijl}(t)\big) - \sum_{j=1}^{n}\int_{-\tau_{ij}}^{0} k_{ij}(t,s)x_j(t+s)\,ds\Bigg), \qquad i = 1, 2, \dots, n, \quad (16)$$

where for all $i, j = 1, 2, \dots, n$ and $L = 1, 2, \dots, l_{ij}$ we assume that

($H_1$) $a_i(t)$, $b_i(t)$, and $a_{ijl}(t)$ are continuous almost periodic functions; $b_i(t) \ge 0$ and $a_{ijl}(t) \ge 0$ for all $t \in R$.

$(\mathrm{H}_2)$ $k_{ij}(t,s)$ is continuous almost periodic with respect to $t \in R$ and is integrable with respect to $s \in [-\tau_{ij}, 0]$; $k_{ij}(t,s) \geq 0$ for all $(t,s) \in R \times [-\tau_{ij}, 0]$ and $\sup\{\int_{-\tau_{ij}}^{0} k_{ij}(t,s)\, ds : t \in R\} < \infty$.

$(\mathrm{H}_3)$ $\tau_{ijl}(t)$ is a nonnegative, almost periodic, and continuous function; $\tau_{ij}$ is a nonnegative constant or $\tau_{ij} = \infty$.

Compared with system (1) we see that

$$f_i(t, x_t) = \sum_{j=1}^{n} \sum_{l=1}^{l_{ij}} a_{ijl}(t) x_j\big(t - \tau_{ijl}(t)\big) + \sum_{j=1}^{n} \int_{-\tau_{ij}}^{0} k_{ij}(t,s) x_j(t+s)\, ds,$$

$$i = 1, 2, \dots, n.$$

Since each $f_i(t, \phi)$ is linear with respect to $\phi \in C^n[-\tau, 0]$, where $\tau = \max\{\tau_{ijl}(t), \tau_{ij} : t \in R,\ i, j = 1, 2, \dots, n,\ l = 1, 2, \dots, l_{ij}\}$. By the assumptions $(\mathrm{H}_1)$–$(\mathrm{H}_3)$ we obtain that $f_i(t, \phi)$ is locally Lipschitz with respect to $\phi \in C^n[-\tau, 0]$ and

$$0 \leq f_i(t, \psi_t) \leq f_i(t, x_{0t}), \qquad \text{for all } \psi \in G, \quad t \in R, \quad i = 1, 2, \dots, n,$$

where $\psi_t(s) = \psi(t+s)$ and $x_{0t}(s) = x_0(t+s)$ for all $s \in [-\tau, 0]$. The function $x_0(t)$ and the set $G$ are defined as above in Section 3. Therefore, by Theorem 1 we obtain the following result.

THEOREM 2. *Under the assumptions* $(\mathrm{H}_1)$–$(\mathrm{H}_3)$, *if* $m(b_i(t)) > 0$ *and*

$$m\left( a_i(t) - \sum_{j=1}^{n} \sum_{l=1}^{l_{ij}} a_{ijl}(t) x_{j0}\big(t - \tau_{ijl}(t)\big) - \sum_{j=1}^{n} \int_{-\tau_{ij}}^{0} k_{ij}(t,s) x_{j0}(t+s)\, ds \right) > 0$$

*for* $i = 1, 2, \dots, n$, *then system* (16) *has at least a positive almost periodic solution.*

Let $f(t)$ be a bounded function on $R$; we denote $f_m = \sup\{f(t) : t \in R\}$ and $f_l = \inf\{f(t) : t \in R\}$.

If $a_i(t) > 0$ and $b_i(t) > 0$ for all $t \in R$ and $(a_i/b_i)_m < \infty$, then directly from Eq. (11) we can obtain that $x_{i0}(t) \leq (a_i/b_i)_m$ for all $t \in R$. Let

$$d_{ij}(t) = \sum_{l=1}^{l_{ij}} a_{ijl}(t) + \int_{-\tau_{ij}}^{0} k_{ij}(t,s)\, ds, \qquad i, j = 1, 2, \dots, n.$$

We have the following corollary of Theorem 2.

COROLLARY 1. *Under the assumptions* $(\mathrm{H}_1)$–$(\mathrm{H}_3)$, *if* $a_i(t) > 0$ *and* $b_i(t) > 0$ *for all* $t \in R$, $(a_i/b_i)_m < \infty$, *and*

$$m\left(a_i(t) - \sum_{j=1}^{n} d_{ij}(t)\left(\frac{a_j}{b_j}\right)_m\right) > 0$$

*for* $i = 1, 2, \ldots, n$, *then system* (16) *has at least a positive almost periodic solution*.

Further, by [3, Theorem 1] (global asymptotical stability) we can obtain the following result.

THEOREM 3. *Suppose that*

(a) *all the conditions of Theorem* 2 *are satisfied*;

(b) $\tau_{ijl}(t)$ $(i, j = 1, 2, \ldots, n,\ l = 1, 2, \ldots, l_{ij})$ *are continuous differentiable and* $\dot{\tau}_{ijl}(t) := d\tau_{ijl}(t)/dt < 1$ *for all* $t \in R$;

(c) *there are positive constants* $T$, $\delta$, *and* $c_i$ $(i = 1, 2, \ldots, n)$ *such that*

$$c_i b_i(t) - \sum_{j=1}^{n} c_j \left( \sum_{l=1}^{l_{ji}} \frac{a_{jil}\left(\sigma_{jil}^{-1}(t)\right)}{1 - \dot{\tau}_{jil}\left(\sigma_{jil}^{-1}(t)\right)} + \int_{-\tau_{ji}}^{0} k_{ji}(t - s, s)\, ds \right) > \delta$$

*for all* $t \geq T$ *and* $i = 1, 2, \ldots, n$, *where* $\sigma_{jil}^{-1}(t)$ *is the inverse function of* $\sigma_{jil}(t) = t - \tau_{jil}(t)$.

*Then system* (16) *has a unique positive almost periodic solution which is globally asymptotical stable*.

Finally, we have the following corollary of Theorem 3.

COROLLARY 2. *Under the assumptions* $(\mathrm{H}_1)$–$(\mathrm{H}_3)$, *if*

(a) $\tau_{ijl}(t) \equiv \tau_{ijl}$ $(i, j = 1, 2, \ldots, n,\ l = 1, 2, \ldots, l_{ij})$ *are constants*, $a_i(t) > 0$ *and* $b_i(t) > 0$ *for all* $t \in R$, *and* $i = 1, 2, \ldots, n$;

(b) $a_{il} > \sum_{j=1}^{n} d_{ijm}(a_j/b_j)_m$, $\forall i = 1, 2, \ldots, n$;

*then system* (16) *has a unique positive almost periodic solution which is globally asymptotical stable*.

In fact, by the conditions (a) and (b) we easily prove that the conditions of Corollary 1 hold. On the other hand, in accordance with the method given in [1, 9] we can obtain that there are positive constants $c_i$ $(i = 1, 2, \ldots, n)$ such that

$$c_i b_{il} - \sum_{j=1}^{n} c_j d_{jim} > 0, \qquad i = 1, 2, \ldots, n.$$

It shows that condition (c) of Theorem 3 holds. Therefore, Corollary 2 is true.

*Remark* 2. Applying the same method given in this paper we can also study the following more general almost periodic Lotka–Volterra systems with delays,

$$\frac{dx_i(t)}{dt} = x_i(t)\big(h_i(t, x_i(t)) - f_i(t, x_t)\big), \qquad i = 1, 2, \dots, n$$

and its special case

$$\frac{dx_i(t)}{dt} = x_i(t)\Bigg(a_i(t) - b_i(t)x_i(t) - \sum_{j=1}^{n}\sum_{l=1}^{l_{ij}} a_{ijl}(t) f_{ijl}\big(x_j(t - \tau_{ijl}(t))\big)$$
$$- \sum_{j=1}^{n}\int_{-\tau_{ij}}^{0} k_{ij}(t, s) g_{ij}\big(x_j(t + s)\big)\, ds\Bigg), \qquad i = 1, 2, \dots, n.$$

We can obtain the results which are similar to Theorem 2 and Corollary 1.

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